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Electro-magnetic wave propagation in more complex linear materials such as bi-anisotropic media have come to a considerable attention within the last fifteen to twenty years. The Drude-Born-Fedorov model has been extensively studied mostly in the time-harmonic case as a model for chiral media. In the physically relevant time-dependent case the record is much less convincing. In this paper we focus on this case and analyze the Drude-Born-Fedorov model in the light of recently developed Hilbert space approach to evolutionary problems. The solution theory will be developed in the framework of extrapolation spaces (Sobolev lattices).

Keywords and phrases: Maxwell's equations, bi-anisotropic material, chiral media, extrapolation spaces, Sobolev lattices

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0 Introduction

The evolutionary system of the macroscopic Maxwell's equations

$$\begin{aligned}\operatorname{curl} H - \partial_0 D &= j, \\ \operatorname{curl} E + \partial_0 B &= 0,\end{aligned}$$

complemented by suitable boundary conditions such as the vanishing of the tangential component of the electric field E and initial data for D and B is completed by a material law connecting the combined six component vector field (D, B) composed from the electric displacement current D and the magnetic induction B with the electro-magnetic field (E, H) (∂_0 denotes the time derivative). In the simplest case it is assumed that

$$\begin{aligned}D &= \varepsilon E \\ B &= \mu H\end{aligned}$$

with constant coefficient $\varepsilon, \mu \in \mathbb{R}_{>0}$ (permittivity ε , permeability μ). This case is referred to as the case of isotropic and homogeneous media. There is a long history of various description of more complex electromagnetic media and a shorter one for their mathematical treatment. As – in a sense – the conclusion of the treatment of the anisotropic and inhomogeneous media case we refer to the presentation of the Maxwell system in the book by R. Leis [9] from 1986 and the references given there. The functional analytical approach utilized shows that indeed a mathematical solution theory can be obtained for rather general media merely requiring that ε, μ are time-independent, bounded, self-adjoint, strictly positive mappings in L^2 -type spaces of vector fields in an non-empty open set $\Omega \subseteq \mathbb{R}^3$ containing the media characterized by ε and μ , which covers the case of real multiplicative matrix-valued operations with in Ω L^∞ -bounded entries as a special case. Although, the necessary concepts are – as one says – well-known since the late seventies of last century, it seems not too widely adopted that by appropriately generalizing the concept of assumption of boundary conditions boundary singularities such as corners, edges, cusps, even fractal boundaries can be included. The price to be paid for this is a more subtle approach exploiting the full power of functional analytic concepts (as a general reference for functional analytical concepts see [7, 23]). As it turns out, however, this can be achieved purely in a Hilbert space setting, thus reducing conveniently the conceptual complexity of our considerations.

Writing $\langle \cdot | \cdot \rangle_0$ for an L^2 -type inner product regardless of the number of components, we have

$$\langle E | H \rangle_0 = \sum_{k=1}^3 \langle E_k | H_k \rangle_0$$

for vector fields

$$E = (E_1, E_2, E_3) = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad H = (H_1, H_2, H_3) = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$$

in $L^2(\Omega)^3$. For ease of notation we shall rarely note the number of components and write e.g. $L^2(\Omega)$ instead of $L^2(\Omega)^3$, since the number of components will always be clear from the context. It is clear from the arguments e.g. in [13], although the impact and relevance of this observation for the more recent discussion of electromagnetic meta-materials could not be realized at the time, that the mathematical methods developed in connection with anisotropic, inhomogeneous media extend to the case of material laws of the block matrix form

$$\begin{pmatrix} D \\ B \end{pmatrix} = M_0 \begin{pmatrix} E \\ H \end{pmatrix}$$

with

$$M_0 = \begin{pmatrix} \varepsilon & \beta \\ \beta^* & \mu \end{pmatrix}$$

in an obvious way, provided that the entries ε, μ, β are time-independent, bounded, linear mappings in L^2 -type spaces of vector fields such that the block matrix operator

$$M_0 : L^2(\Omega) \rightarrow L^2(\Omega) \\ \begin{pmatrix} E \\ H \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon & \beta \\ \beta^* & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}$$

is still self-adjoint and strictly positive.¹

Based on a structural observation described in [17] we shall here consider in the following a class of well-posed evolutionary equations in even more complex media. Such media have become of strong interest in recent decades due to their prospects in view of their interesting physical properties (for a survey we refer the reader to [11, 8]). This problem class, which conveniently contains materials with memory, will lead us to the so-called Drude-Born-Fedorov model for chiral media, which has been initially introduced as an approximation of the actual material description in an attempt to avoid non-local-in-time terms (optical response).

The model starts from a material relation of the form

$$D = \varepsilon(E + \eta \operatorname{curl} E), \quad B = \mu(H + \eta \operatorname{curl} H),$$

where $\varepsilon, \mu \in \mathbb{R}_{>0}$ and $\eta \in \mathbb{R}$ is the so-called chirality parameter.

There is a long list of publications dealing with this model. Despite this fact and contrary to various claims made, it appears that this model has not been fully understood in sufficient

¹This is the case if and only if $\varepsilon, \mu - \beta^* \varepsilon^{-1} \beta$ are selfadjoint and strictly positive, which amounts to a smallness condition on the off-diagonal entry β .

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generality. It is due to this state of affairs that we have chosen to speak of the “elusive” Drude-Born-Fedorov model. So has it only more recently been understood that the model has to be considered in the – much less forgiving – dynamic case, [6, 22, 1, 2, 3], rather than the so-called time-harmonic case.

We will show that in the framework of our approach from [17] the Drude-Born-Fedorov model will have to be considered as a rather degenerate case, which, however, by some additional considerations can be elegantly embedded into the standard theory. Indeed, we are led to a simple evolution equation with a bounded generator. This has in essence already been observed in [10] for simply connected domains with smooth boundary. It has gone unnoticed, however, that the result can be extended to fairly arbitrary open sets with a boundary permitting a suitable local compact embedding result, [14, 15, 5]. As a by-product we can discuss well-posedness for a class of bi-anisotropic materials with *modified* Drude-Born-Fedorov type material behavior, some cases of which have found previous attention, compare [12, Morro 2002], [21, Sjöberg 2008].

1 Space-Time Evolution Equations

1.1 A brief summary of Sobolev chains and lattices

We recall first the standard construction of chains of Hilbert spaces associated with a normal operator O in a Hilbert space X with 0 in its resolvent set, see e.g. [18]. Defining $H_k(O)$ as the completion of $D(O^k)$ with respect to the norm $|\cdot|_k$ induced by the inner product $\langle \cdot | \cdot \rangle_k$ given by

$$(f, g) \mapsto \langle O^k f | O^k g \rangle_X$$

for $k \in \mathbb{Z}$, we obtain a chain $(H_k(O))_{k \in \mathbb{Z}}$ of Hilbert space, where the order is given by canonical, continuous and dense embeddings

$$H_{k+1}(O) \hookrightarrow H_k(O), \quad k \in \mathbb{Z}.$$

Note that $H_1(O)$ is the domain of O and for $k \in \mathbb{N}$

$$H_k(O) \hookrightarrow X \hookrightarrow H_{-k}(O)$$

is a Gelfand triple. By construction

$$\begin{aligned} H_1(O) &\rightarrow H_0(O) \\ f &\mapsto Of \end{aligned}$$

is a unitary mapping and it follows by induction that for $k \in \mathbb{N}$ also

$$\begin{aligned} H_{k+1}(O) &\rightarrow H_k(O) \\ f &\mapsto Of \end{aligned}$$

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is unitary. Moreover, O extends to a unitary mapping

$$\begin{aligned} H_{k+1}(O) &\rightarrow H_k(O) \\ f &\mapsto Of \end{aligned}$$

also for negative k and so to all $k \in \mathbb{Z}$.

Remark 1.1. Since $O = \Re O + i \Im O$ is a normal operator in $H_0(O)$ we have

$$O^* = \Re O - i \Im O$$

Since $D(O) = D(O^*)$ and

$$|Of|_X = |O^*f|_X$$

that also O has a unitary extension

$$\begin{aligned} H_{k+1}(O) &\rightarrow H_k(O) \\ f &\mapsto O^*f \end{aligned}$$

for $k \in \mathbb{Z}$. Keeping the notation O^* for this extension, as we shall do, leads to a possible confusion, since O^* could be misinterpreted as the adjoint, i.e. the inverse of O as a unitary mapping. To avoid this misunderstanding we shall *not* write the inverse of a unitary operator as its adjoint.

With this convention we can safely say that we have established O^s and $(O^*)^s$ as unitary mappings

$$\begin{aligned} H_{k+s}(O) &\rightarrow H_k(O) \\ f &\mapsto O^s f \end{aligned}$$

and

$$\begin{aligned} H_{k+s}(O) &\rightarrow H_k(O) \\ f &\mapsto (O^*)^s f \end{aligned}$$

for $s, k \in \mathbb{Z}$.

We shall utilize this abstract construction for various special cases. Note that for $O \in L(H, H)$ we have

$$H_s(O) = H$$

as topological linear spaces with merely different inner products (inducing equivalent norms) in the different Hilbert spaces $H_s(O)$, $s \in \mathbb{Z}$. This indicates that considering continuous linear operators O does not lead to interesting chains.

A particular instance of this construction is if O is the time-derivative ∂_0 . We recall, e.g. from [17, 18], that differentiation considered in the complex Hilbert space $H_{\varrho,0}(\mathbb{R}) := \{f \in L^2_{\text{loc}}(\mathbb{R}) | (x \mapsto \exp(-\varrho x)f(x)) \in L^2(\mathbb{R})\}$, $\varrho \in \mathbb{R} \setminus \{0\}$, with inner product

$$(f, g) \mapsto \langle f, g \rangle_{\varrho,0} := \int_{\mathbb{R}} f(x)^* g(x) \exp(-2\varrho x) dx$$

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can indeed be established as a normal operator, which we denote by $\partial_{0,\varrho}$, with

$$\Re \partial_{0,\varrho} = \varrho.$$

For $\Im \partial_{0,\varrho}$ we have as a spectral representation the Fourier-Laplace transform $\mathcal{L}_\varrho : H_{\varrho,0}(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ given by the unitary extension of

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}) \subseteq H_{\varrho,0}(\mathbb{R}) &\rightarrow L^2(\mathbb{R}) \\ \phi &\mapsto \left(x \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-ixy) \exp(-\varrho y) \phi(y) dy \right). \end{aligned}$$

In other words, we have the unitary equivalence

$$\Im \partial_{0,\varrho} = \mathcal{L}_\varrho^{-1} m \mathcal{L}_\varrho,$$

where m denotes the selfadjoint multiplication-by-argument operator in $L^2(\mathbb{R})$. Since 0 is in the resolvent set of $\partial_{0,\varrho}^{-1}$ we have that $\partial_{0,\varrho}^{-1}$ is an element of the Banach space $L(H_{\varrho,0}(\mathbb{R}), H_{\varrho,0}(\mathbb{R}))$ of continuous (left-total) linear mappings in $H_{\varrho,0}(\mathbb{R})$. Denoting generally the operator norm of the Banach space $L(X, Y)$ by $\|\cdot\|_{L(X,Y)}$, we get for ∂_0^{-1}

$$\|\partial_{0,\varrho}^{-1}\|_{L(H_{\varrho,0}(\mathbb{R}), H_{\varrho,0}(\mathbb{R}))} = \frac{1}{\varrho}.$$

Not too surprisingly, we find for $\varrho > 0$

$$(\partial_{0,\varrho}^{-1} \varphi)(x) = \int_{-\infty}^x \varphi(t) dt$$

and for $\varrho < 0$

$$(\partial_{0,\varrho}^{-1} \varphi)(x) = - \int_x^\infty \varphi(t) dt$$

for all $\varphi \in \mathring{C}_\infty(\mathbb{R})$ and $x \in \mathbb{R}$. Since we are interested in the forward causal situation, we assume $\varrho > 0$ throughout. Moreover, in the following we shall mostly write ∂_0 for $\partial_{0,\varrho}$ if the choice of ϱ is clear from the context.

Thus, with $O = \partial_{0,\varrho}$ in the above general construction we obtain a chain $(H_{\varrho,k}(\mathbb{R}))_{k \in \mathbb{Z}}$ of Hilbert spaces

$$H_{\varrho,k}(\mathbb{R}) := H_k(\partial_{0,\varrho}).$$

Similarly, with $O = im + \varrho$ in $L^2(\mathbb{R})$ we obtain the chain of polynomially weighted $L^2(\mathbb{R})$ -spaces

$$(L_k^2(\mathbb{R}))_{k \in \mathbb{Z}}$$

with

$$L_k^2(\mathbb{R}) := \left\{ f \in L^{2,\text{loc}}(\mathbb{R}) \mid (im + \varrho)^k f \in L^2(\mathbb{R}) \right\} = H_k(im + \varrho)$$

for $k \in \mathbb{Z}$.

1.1 A brief summary of Sobolev chains and lattices

Since the unitarily equivalent operators $\partial_{0,\varrho}$ and $im + \varrho$ can canonically be lifted to the X -valued case, X an arbitrary complex Hilbert space, we are lead to a corresponding chain $(H_{\varrho,k}(\mathbb{R}, X))_{k \in \mathbb{Z}}$ and $(L_k^2(\mathbb{R}, X))_{k \in \mathbb{Z}}$ of X -valued generalized functions. The Fourier-Laplace transform can also be lifted to the X -valued case yielding

$$\begin{aligned} H_{\varrho,k}(\mathbb{R}, X) &\rightarrow L_k^2(\mathbb{R}, X) \\ f &\mapsto \mathcal{L}_\varrho f \end{aligned}$$

as a unitary mapping for $k \in \mathbb{N}$ and by continuous extension, keeping the notation \mathcal{L}_ϱ for the extension, also for $k \in \mathbb{Z}$. Since \mathcal{L}_ϱ has been constructed from a spectral representation of $\Im m \partial_{0,\varrho}$, we can utilize the corresponding operator function calculus for functions of $\Im m \partial_{0,\varrho}$. Noting that $\partial_{0,\varrho} = i \Im m \partial_{0,\varrho} + \varrho$ is a function of $\Im m \partial_{0,\varrho}$ we can define operator-valued functions of ∂_0 .

Definition 1.2. Let $r > \frac{1}{2\varrho} > 0$ and $M : B_{\mathbb{C}}(r, r) \rightarrow L(H, H)$ be bounded and analytic, H a Hilbert space. Then define

$$M(\partial_0^{-1}) := \mathbb{L}_\varrho^* M\left(\frac{1}{im + \varrho}\right) \mathbb{L}_\varrho,$$

where

$$M\left(\frac{1}{im + \varrho}\right) \phi(t) := M\left(\frac{1}{it + \varrho}\right) \phi(t) \quad (t \in \mathbb{R})$$

for $\phi \in \mathring{C}_\infty(\mathbb{R}, H)$.

Remark 1.3. The definition of $M(\partial_0^{-1})$ is largely independent of the choice of ϱ in the sense that the operators for two different parameters ϱ_1, ϱ_2 coincide on the intersection of the respective domains.

Simple examples are polynomials in ∂_0^{-1} with operator coefficients. A more exotic example of an analytic and bounded function of ∂_0^{-1} is the delay operator, which itself is a special case of the time translation:

Example 1.4. Let $r > 0$, $\varrho > \frac{1}{2r}$, $h \in \mathbb{R}$ and $u \in H_{\varrho,0}(\mathbb{R}, X)$. We define

$$\tau_h u := u(\cdot + h).$$

The operator $\tau_h \in L(H_{\varrho,0}(\mathbb{R}, X), H_{\varrho,0}(\mathbb{R}, X))$ is called a *time-translation operator*. If $h < 0$ the operator τ_h is also called a *delay operator*. In the latter case the function

$$B_{\mathbb{C}}(r, r) \ni z \mapsto M(z) := \exp(z^{-1}h)$$

is analytic and uniformly bounded for every $r \in \mathbb{R}_{>0}$ (considered as an $L(X, X)$ -valued function). An easy computation shows for $u \in H_{\varrho,0}(\mathbb{R}, X)$ that

$$u(\cdot + h) = \mathbb{L}_\varrho^* \exp((im + \varrho)h) \mathbb{L}_\varrho u = M(\partial_0^{-1})u = \exp((\partial_0^{-1})^{-1}h)u.$$

This shows that

$$\tau_h = \exp(h/\partial_0^{-1}) = \exp(h\partial_0).$$

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Another class of interesting bounded analytic functions of ∂_0^{-1} are mappings produced by a temporal convolution with a suitable operator-valued integral kernel.

Let now O denote a normal operator in Hilbert space H with 0 in the resolvent set. Then O has a canonical extension to the time-dependent case i.e. to $H_{\varrho,0}(\mathbb{R}, H)$. Then ∂_0 and O become commuting normal operators and by combining the two chains we obtain a Sobolev lattice in the sense of [19] based on (∂_0, O) yielding a family of Hilbert spaces $(\varrho \in \mathbb{R} \setminus \{0\})$

$$(H_{\varrho,k}(\mathbb{R}, H_s(O)))_{k,s \in \mathbb{Z}}$$

with norms $|\cdot|_{\varrho,k,s}$ given by

$$v \mapsto |\partial_0^k O^s v|_{H_{\varrho,0}(\mathbb{R}, H)}$$

for $k, s \in \mathbb{Z}$. Note that

$$H_0(O) = H$$

independent of the particular choice of O .

1.2 Abstract initial value problems

We shall discuss equations of the form

$$(\partial_0 M(\partial_0^{-1}) + A)U = \mathcal{J} + \delta \otimes W_0, \quad (1)$$

where for simplicity we shall assume that A is skew-selfadjoint in a Hilbert space H and M is a regular material law in the sense of [16, 19]. More specifically we assume that M is of the form

$$M(z) = M_0 + zM_1(z)$$

where M_1 is an analytic and bounded $L(H, H)$ -valued function in a ball $B_{\mathbb{C}}(r, r)$ for some $r \in \mathbb{R}_{>0}$ and M_0 is a continuous, selfadjoint and strictly positive definite operator in H . The operator $M(\partial_0^{-1})$ is then to be understood in the sense of the operator-valued function calculus associated with the selfadjoint operator $\mathfrak{Im}(\partial_0) = \frac{1}{2i}(\partial_0 + \partial_0^*)$.

For the data we assume

$$\mathcal{J} \in H_{\varrho,0}(\mathbb{R}, H), \quad \mathcal{J} = 0 \text{ on } \mathbb{R}_{<0},$$

and

$$W_0 \in H,$$

which makes (1) an abstract initial value problem. The appropriate setting turns out to be the Sobolev lattice

$$\left(H_{\varrho,k} \left(\mathbb{R}, H_s \left(\sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} + 1 \right) \right) \right)_{k,s \in \mathbb{Z}},$$

where, however, only the spaces with $s = -1, 0, 1$ and $k = -2, -1, 0, 1$ are actually utilized. From [16, 19] we paraphrase the following solution result on which our approach to the Drude-Born-Fedorov model can conveniently be based.

1.2 Abstract initial value problems

Theorem 1.5. *The abstract initial value problem (1) has a unique solution $U \in H_{\varrho,-1}(\mathbb{R}, H)$. Moreover,*

$$F \mapsto (\partial_0 M (\partial_0^{-1}) + A)^{-1} F$$

is a linear mapping in $L(H_{\varrho,k}(\mathbb{R}, H), H_{\varrho,k}(\mathbb{R}, H))$, $k \in \mathbb{Z}$. These mappings are causal in the sense that if $F \in H_{\varrho,k}(\mathbb{R}, H)$ vanishes on the time interval $] -\infty, a]$, then so does $(\partial_0 M (\partial_0^{-1}) + A)^{-1} F$, $a \in \mathbb{R}$, $k \in \mathbb{Z}$. In particular, we have

$$U = 0 \text{ on } \mathbb{R}_{<0}.$$

To link up with a more classical interpretation of the assumption of the initial data we record the following regularity result.

Theorem 1.6. *Let $U \in H_{\varrho,-1}(\mathbb{R}, H)$ be the unique solution of the abstract initial value problem (1). Then, we also have*

$$U \in H_{\varrho,0}(\mathbb{R}, H) \tag{2}$$

and U is a continuous² H -valued function on $\mathbb{R} \setminus \{0\}$. Moreover,

$$U(0+) = M_0^{-1} W_0, \tag{3}$$

where the limit is taken in H .

Proof. These stronger regularity statements will rely on the function calculus for the skew-selfadjoint operator $\sqrt{M_0^{-1}} A \sqrt{M_0^{-1}}$ or – depending on the point of view – on one-parameter semi-group arguments. Let $U \in H_{\varrho,-1}(\mathbb{R}, H)$ be the solution of (1). Noting that

$$(\partial_0 M_0 + A)^{-1} = \sqrt{M_0^{-1}} \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \sqrt{M_0^{-1}}$$

we have

$$\left(\partial_0 + \sqrt{M_0^{-1}} M_1 (\partial_0^{-1}) \sqrt{M_0^{-1}} + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right) \sqrt{M_0} U = \sqrt{M_0^{-1}} \mathcal{J} + \delta \otimes \sqrt{M_0^{-1}} W_0.$$

From this we have

$$\begin{aligned} \left(1 + \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} M_1 (\partial_0^{-1}) \right) \sqrt{M_0} U &= \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \sqrt{M_0^{-1}} \mathcal{J} + \\ &+ \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \delta \otimes \sqrt{M_0^{-1}} W_0. \end{aligned}$$

²In the usual sense of having a continuous representer.

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We have

$$\left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \delta \otimes \sqrt{M_0^{-1}} W_0 = \left(t \mapsto \chi_{\mathbb{R}_{\geq 0}}(t) \exp \left(-t \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right) \sqrt{M_0^{-1}} W_0 \right)$$

by uniqueness of solution, which as we read off is in $H_{\varrho,0}(\mathbb{R}, H)$ and is a continuous function on $\mathbb{R} \setminus \{0\}$. In particular, we also have

$$\left(\left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \delta \otimes \sqrt{M_0^{-1}} W_0 \right) (0+) = \sqrt{M_0^{-1}} W_0 \quad (4)$$

in H . Moreover, for $\varrho \in \mathbb{R}_{>0}$ sufficiently large we have that $\left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} M_1 (\partial_0^{-1})$ is a contraction since

$$\left| \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} f \right|_{\varrho,0,0} \leq \frac{1}{\varrho} |f|_{\varrho,0,0}.$$

Therefore

$$U = \sqrt{M_0^{-1}} (1 - Q)^{-1} \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \left(\sqrt{M_0^{-1}} \mathcal{J} + \delta \otimes \sqrt{M_0^{-1}} W_0 \right) \in H_{\varrho,0}(\mathbb{R}, H),$$

where $Q = - \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} M_1 (\partial_0^{-1})$. We also have that

$$\begin{aligned} & \left(1 + \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} M_1 (\partial_0^{-1}) \right)^{-1} = \\ & = 1 - \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} M_1 (\partial_0^{-1}) \left(1 + \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} M_1 (\partial_0^{-1}) \right)^{-1} \end{aligned}$$

and so

$$\begin{aligned} U &= \sqrt{M_0^{-1}} \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} \delta \otimes \sqrt{M_0^{-1}} W_0 + \\ &+ \sqrt{M_0^{-1}} \left(\partial_0 + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right)^{-1} F \end{aligned} \quad (5)$$

for some $F \in H_{\varrho,0}(\mathbb{R}, H)$ with $F = 0$ on $\mathbb{R}_{<0}$. Since

$$\begin{aligned} \left| \int_{-\infty}^t \exp \left(- (t-s) \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right) F(s) ds \right| &\leq \int_{-\infty}^t |F(s)| ds \\ &= \int_{-\infty}^t \exp(\varrho s) |F(s)| \exp(-\varrho s) ds, \\ &\leq \frac{1}{\sqrt{2\varrho}} \exp(\varrho t) \sqrt{\int_{-\infty}^t |F(s)|^2 \exp(-2\varrho s) ds}, \end{aligned}$$

existence of the integral $\int_{-\infty}^t \exp\left(-(t-s)\sqrt{M_0^{-1}}A\sqrt{M_0^{-1}}\right)F(s)ds$ and continuity of $t \mapsto \int_{-\infty}^t \exp\left(-(t-s)\sqrt{M_0^{-1}}A\sqrt{M_0^{-1}}\right)F(s)ds$ is clear. Since $F = 0$ on $\mathbb{R}_{<0}$ we have in particular that

$$\int_{-\infty}^0 \exp\left(-(t-s)\sqrt{M_0^{-1}}A\sqrt{M_0^{-1}}\right)F(s)ds = 0.$$

Since the first term of (5) satisfies property (4) we have indeed

$$U(0+) = \sqrt{M_0^{-1}}\sqrt{M_0^{-1}}W_0 = M_0^{-1}W_0.$$

□

Remark 1.7. In the case $A = 0$ even stronger regularity follows. Indeed, the solution is then such that

$$U - \chi_{\mathbb{R}_{\geq 0}} \otimes M_0^{-1}W_0 \in H_{\ell,1}(\mathbb{R}, H). \quad (6)$$

2 Discussion of the Drude-Born-Fedorov Model

We begin by analyzing the Drude-Born-Fedorov material relation $(\varepsilon, \mu \in \mathbb{R}_{>0}, \eta \in \mathbb{R} \setminus \{0\})$

$$D = \varepsilon(E + \eta \operatorname{curl} E), \quad B = \mu(H + \eta \operatorname{curl} H)$$

in the light of the above general theory by substituting Maxwell's equations back into the Drude-Born-Fedorov relation (ignoring possible source terms) to obtain a modified material relation of the form

$$D = \varepsilon(E - \eta \partial_0 B), \quad B = \mu(H + \eta \partial_0 D).$$

This is a modified Condon model, [4, Condon 1937], compared to which in the right-hand sides B and D are replaced by H and E , respectively.

Although similar in concept to the Condon model, where replacing a convolution term by a two-term Taylor approximation leads to no reasonable results, here this type of “approximation” can actually be justified, see [6, Frantzeskakis, Ioannidis, Roach, Stratis, Yannacopoulos (2003)]. With a slight reformulation we have Maxwell's equations

$$\partial_0 \begin{pmatrix} \frac{1}{\sqrt{\varepsilon}} D \\ \frac{1}{\sqrt{\mu}} B \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\mu}} E \\ \frac{1}{\sqrt{\varepsilon}} H \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{\varepsilon}} J \\ 0 \end{pmatrix}$$

and the Drude-Born-Fedorov material relation assumes the formal shape

$$\begin{pmatrix} 1 & \eta \sqrt{\varepsilon \mu} \partial_0 \\ -\eta \sqrt{\varepsilon \mu} \partial_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\varepsilon}} D \\ \frac{1}{\sqrt{\mu}} B \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\mu}} E \\ \frac{1}{\sqrt{\varepsilon}} H \end{pmatrix},$$

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or rather

$$\begin{aligned} \begin{pmatrix} \frac{1}{\sqrt{\varepsilon}} D \\ \frac{1}{\sqrt{\mu}} B \end{pmatrix} &= \begin{pmatrix} 1 & \eta \sqrt{\varepsilon \mu} \partial_0 \\ -\eta \sqrt{\varepsilon \mu} \partial_0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{\sqrt{\mu}} E \\ \frac{1}{\sqrt{\varepsilon}} H \end{pmatrix} \\ &= (1 + \eta^2 \varepsilon \mu \partial_0^2)^{-1} \begin{pmatrix} 1 & -\eta \sqrt{\varepsilon \mu} \partial_0 \\ \eta \sqrt{\varepsilon \mu} \partial_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\mu}} E \\ \frac{1}{\sqrt{\varepsilon}} H \end{pmatrix}. \end{aligned}$$

In the language of the above theory

$$\begin{aligned} M(\partial_0^{-1}) &= (1 + \eta^2 \varepsilon \mu \partial_0^2)^{-1} \begin{pmatrix} 1 & -\eta \sqrt{\varepsilon \mu} \partial_0 \\ \eta \sqrt{\varepsilon \mu} \partial_0 & 1 \end{pmatrix} \\ &= \left(1 + \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 \right)^{-1} \begin{pmatrix} \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 & - \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right) \\ \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right) & \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} 0 & -\frac{1}{\eta \sqrt{\varepsilon \mu}} \\ \frac{1}{\eta \sqrt{\varepsilon \mu}} & 0 \end{pmatrix} + \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 \left(1 + \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 \right)^{-1} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} 0 & -\frac{1}{\eta \sqrt{\varepsilon \mu}} \\ \frac{1}{\eta \sqrt{\varepsilon \mu}} & 0 \end{pmatrix} + O(\partial_0^{-2}). \end{aligned}$$

Note

$$\Re \begin{pmatrix} 0 & -\frac{1}{\eta \sqrt{\varepsilon \mu}} \\ \frac{1}{\eta \sqrt{\varepsilon \mu}} & 0 \end{pmatrix} = 0$$

which shows that the resulting equation is *not* covered by the above theory directly and actually is not a differential equation in time at all:

$$\begin{aligned} &\left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 \left(1 + \left(\frac{1}{\eta \sqrt{\varepsilon \mu}} \partial_0^{-1} \right)^2 \right)^{-1} + \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{\sqrt{\mu}} E \\ \frac{1}{\sqrt{\varepsilon}} H \end{pmatrix} = \\ &= \begin{pmatrix} -\frac{1}{\sqrt{\varepsilon}} J \\ \frac{1}{\sqrt{\mu}} 0 \end{pmatrix}. \end{aligned}$$

Moreover, this whole consideration is assuming that we consider curl to be a self-adjoint realization in $L^2(\Omega)$, which can only be achieved in very special cases, such as $\Omega = \mathbb{R}^3 \setminus N$, where N is a set of capacity zero. (e.g. in $\Omega = \mathbb{R}^3$).

However, for media occupying an arbitrary open subset Ω of \mathbb{R}^3 we fortunately have a natural choice of boundary condition, which turns curl with a corresponding choice of domain into a selfadjoint operator. To formulate this condition properly we need to introduce curl

as the closure in $L^2(\Omega)$ of curl restricted to $\mathring{C}_\infty(\Omega)$ vector fields (we do *not* indicate the number of components). Then curl is properly defined as the adjoint of curl :

$$\text{curl} := \left(\mathring{\text{curl}} \right)^*.$$

Similarly, div is defined as the closure in $L^2(\Omega)$ of div restricted to $\mathring{C}_\infty(\Omega)$ vector fields. Then

$$\text{grad} := - \left(\mathring{\text{div}} \right)^*$$

is the usual weak derivative in $L^2(\Omega)$, see [14, πk 1998] for the conceptual details. Containment of a field E in $D\left(\mathring{\text{curl}}\right)$ is the proper weak generalization of the classical boundary condition “ $n \times E = 0$ on $\partial\Omega$ ”, whereas $E \in D\left(\mathring{\text{div}}\right)$ generalized the classical boundary condition “ $n \cdot E = 0$ on $\partial\Omega$ ”. It is important here to keep in mind that no regularity assumptions on the boundary and no trace results are needed for these generalized constructions.

A suitable boundary condition for the Drude-Born-Fedorov model can now be stated in terms of the range of curl , $R\left(\mathring{\text{curl}}\right)$, and of the domain of div , $D\left(\mathring{\text{div}}\right)$.

We require

$$\text{curl } E \in \overline{R\left(\mathring{\text{curl}}\right)} \tag{7}$$

or equivalently

$$\text{curl } E \in D\left(\mathring{\text{div}}\right) \tag{8}$$

and

$$\text{curl } E \perp \mathcal{H}_N, \tag{9}$$

where \mathcal{H}_N denotes the set of harmonic Neumann fields

$$\mathcal{H}_N = \left\{ E \in D\left(\mathring{\text{div}}\right) \mid \text{div } E = 0, \text{curl } E = 0 \right\}.$$

Condition (8) generalizes the classical boundary condition “ $n \cdot \text{curl } E = 0$ on $\partial\Omega$ ” to non-smooth boundaries and data³. We shall denote the operator curl subject to boundary conditions (8), (9) by $\mathring{\text{curl}}$. Under fairly general assumptions it can be shown that $\mathring{\text{curl}}$ is actually selfadjoint and even has – apart from 0 – only discrete spectrum $\sigma_d\left(\mathring{\text{curl}}\right)$,

$$\sigma_p\left(\mathring{\text{curl}}\right) \setminus \{0\} = \sigma_d\left(\mathring{\text{curl}}\right). \tag{10}$$

³Note that assuming data in $D\left(\mathring{\text{div}}\right)$ boundary condition (8) is induced by the requirement that $E \in D\left(\mathring{\text{div}}\right)$. Moreover, in the simply connected case $\mathcal{H}_N = \{0\}$ so that (7) reduces to (8). These observations are the link to the set-up utilized in [10].

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The most general result, [5, Filonov 2000], merely requires Ω to be an open set with bounded measure to obtain the same properties for $\mathring{\text{curl}}$. Exterior domains, i.e. open sets with compact complement, also support selfadjointness of $\mathring{\text{curl}}$, [15, π k 1998], if the boundary satisfies a local compact embedding property, see [20]. Indeed, although in the exterior domain case the spectrum of $\mathring{\text{curl}}$ is not purely point spectrum anymore, we still maintain (10).

In order to avoid technicalities and to immunize the results presented here against possible future improvements, we make the crucial property of selfadjointness our core assumption for the analysis of the Drude-Born-Fedorov model:

Hypothesis Ω : We assume that Ω is an open subset of \mathbb{R}^3 such that

- $\mathring{\text{curl}}$ is selfadjoint.

Under this general assumption we shall re-inspect the Drude-Born-Fedorov model from another perspective. We first note that

$$D = \varepsilon (E + \eta \text{ curl } E) = (1 + \eta \text{ curl}) \varepsilon E, \quad B = \mu (H + \eta \text{ curl } H) = (1 + \eta \text{ curl}) \mu H.$$

So, imposing our general **Hypothesis Ω** the material relation takes on the form

$$\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} 1 + \eta \mathring{\text{curl}} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}.$$

The initial value problem for Maxwell's equation now reads formally

$$\begin{pmatrix} 1 + \eta \mathring{\text{curl}} \end{pmatrix} \partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathring{\text{curl}} \begin{pmatrix} E \\ H \end{pmatrix} = \mathcal{J} + \delta \otimes W_0.$$

Now let P_η denote the orthogonal projector onto $\overline{R \begin{pmatrix} 1 + \eta \mathring{\text{curl}} \end{pmatrix}}$. Consequently, $(1 - P_\eta)$ is the orthogonal projector onto $N \begin{pmatrix} 1 + \eta \mathring{\text{curl}} \end{pmatrix}$ and so

$$\begin{aligned} (1 - P_\eta) \mathring{\text{curl}} &= \eta^{-1} (1 - P_\eta) \left(\eta \mathring{\text{curl}} + 1 - 1 \right) \\ &= -\eta^{-1} (1 - P_\eta) \end{aligned} \tag{11}$$

$$\begin{aligned} \mathring{\text{curl}} (1 - P_\eta) &= \eta^{-1} \left(\eta \mathring{\text{curl}} + 1 - 1 \right) (1 - P_\eta) \\ &= \eta^{-1} \left(\eta \mathring{\text{curl}} + 1 \right) (1 - P_\eta) - \eta^{-1} (1 - P_\eta) \\ &= -\eta^{-1} (1 - P_\eta). \end{aligned}$$

Thus, P_η and $\overset{\diamond}{\text{curl}}$ commute and we have

$$\begin{aligned}
& \left(1 + \eta \overset{\diamond}{\text{curl}}\right) \partial_0 P_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \\
& \quad + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_\eta \overset{\diamond}{\text{curl}} \begin{pmatrix} E \\ H \end{pmatrix} = P_\eta \mathcal{J} + \delta \otimes P_\eta W_0 \\
& \left(1 + \eta \overset{\diamond}{\text{curl}}\right) \partial_0 P_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} P_\eta \begin{pmatrix} E \\ H \end{pmatrix} + \\
& + \left(1 + \eta \overset{\diamond}{\text{curl}}\right) \partial_0 P_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} (1 - P_\eta) \begin{pmatrix} E \\ H \end{pmatrix} + \\
& \quad + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_\eta \overset{\diamond}{\text{curl}} \begin{pmatrix} E \\ H \end{pmatrix} = P_\eta \mathcal{J} + \delta \otimes P_\eta W_0 \\
& \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1 - P_\eta) \overset{\diamond}{\text{curl}} \begin{pmatrix} E \\ H \end{pmatrix} = (1 - P_\eta) \mathcal{J} + \delta \otimes (1 - P_\eta) W_0 \\
& -\eta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1 - P_\eta) \begin{pmatrix} E \\ H \end{pmatrix} = 0
\end{aligned}$$

and so

$$\begin{aligned}
& \left(1 + \eta \pi_\eta \overset{\diamond}{\text{curl}} \pi_\eta^*\right) \partial_0 \pi_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \pi_\eta^* \pi_\eta \begin{pmatrix} E \\ H \end{pmatrix} + \\
& \quad + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pi_\eta \overset{\diamond}{\text{curl}} \pi_\eta^* \pi_\eta \begin{pmatrix} E \\ H \end{pmatrix} = \pi_\eta \mathcal{J} + \delta \otimes \pi_\eta W_0 \\
& \partial_0 \left(\pi_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \pi_\eta^* \right) \pi_\eta \begin{pmatrix} E \\ H \end{pmatrix} + \\
& + \left(1 + \eta \pi_\eta \overset{\diamond}{\text{curl}} \pi_\eta^*\right)^{-1} \pi_\eta \overset{\diamond}{\text{curl}} \pi_\eta^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pi_\eta \begin{pmatrix} E \\ H \end{pmatrix} = \pi_\eta \mathcal{J} + \delta \otimes \pi_\eta W_0
\end{aligned}$$

It is

$$P_\eta \overset{\diamond}{\text{curl}} \subseteq \overset{\diamond}{\text{curl}} P_\eta$$

or

$$\left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \left(1 + \eta \overset{\diamond}{\text{curl}}\right)^{-1} \overset{\diamond}{\text{curl}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \right) = \left(1 + \eta \overset{\diamond}{\text{curl}}\right)^{-1} (\mathcal{J} + \delta \otimes W_0).$$

Discussing this evolution equation we can finally make rigorous sense of our above arguments.

The solution theory of the original Drude-Born-Fedorov model can now be formulated within the framework of the Sobolev lattice

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$$\left(H_{\varrho,k} \left(\mathbb{R}, H_s \left(\overset{\diamond}{\text{curl}} + i \right) \right) \right)_{k,s \in \mathbb{Z}}.$$

Theorem 2.1. *Under **Hypothesis Ω** and assuming that*

$$-\frac{1}{\eta} \in \left(\rho \left(\overset{\diamond}{\text{curl}} \right) \cap \mathbb{R} \right) \cup \left\{ \lambda \mid \lambda \text{ is isolated in } \sigma \left(\overset{\diamond}{\text{curl}} \right) \right\} \quad (12)$$

we have that for

$$\mathcal{J} \in H_{\varrho,0} \left(\mathbb{R}, R \left(1 + \eta \overset{\diamond}{\text{curl}} \right) \right), \quad \mathcal{J} = 0 \text{ on } \mathbb{R}_{<0}, \quad (13)$$

and

$$W_0 \in R \left(1 + \eta \overset{\diamond}{\text{curl}} \right) \quad (14)$$

the Drude-Born-Fedorov model

$$\partial_0 \begin{pmatrix} D \\ B \end{pmatrix} + \begin{pmatrix} 0 & -\overset{\diamond}{\text{curl}} \\ \overset{\diamond}{\text{curl}} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathcal{J} + \delta \otimes W_0 \quad (15)$$

with

$$\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} 1 + \eta \overset{\diamond}{\text{curl}} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \quad (16)$$

has a unique solution $\begin{pmatrix} E \\ H \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega))$ satisfying

$$\begin{pmatrix} D \\ B \end{pmatrix} (0+) = W_0$$

in $H_{-1} \left(\overset{\diamond}{\text{curl}} + i \right)$. The solution depends continuously and causal on the data.

Proof. We have that

$$\overset{\diamond}{\text{curl}}_{\eta} : D \left(\overset{\diamond}{\text{curl}} \right) \cap \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)} \subseteq \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)} \rightarrow R \left(1 + \eta \overset{\diamond}{\text{curl}} \right) \subseteq \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)}$$

$$\phi \mapsto \overset{\diamond}{\text{curl}} \phi$$

is a selfadjoint operator with a well-defined continuous inverse

$$\left(1 + \eta \overset{\diamond}{\text{curl}}_{\eta} \right)^{-1} : \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)} \rightarrow R \left(1 + \eta \overset{\diamond}{\text{curl}} \right) \subseteq \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)}.$$

It is

$$\pi_\eta \overset{\diamond}{\text{curl}} \pi_\eta^* = \overset{\diamond}{\text{curl}}_\eta$$

where π_η denotes the canonical projection onto the subspace $\overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)}$ of $L^2(\Omega)$.

Then π_η^* is the identity embedding of $\overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)}$ into $L^2(\Omega)$.

We consider initially the solution of the evolution equation

$$\begin{aligned} & \left(\partial_0 \pi_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \pi_\eta^* + \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \overset{\diamond}{\text{curl}}_\eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} e \\ h \end{pmatrix} = \\ & = \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \mathcal{J} + \\ & + \delta \otimes \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} W_0 \end{aligned} \quad (17)$$

and observe that the resulting system is now an evolutionary system in the sense of Section 1 with

$$M(\partial_0^{-1}) = \pi_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \pi_\eta^* + \partial_0^{-1} C, \quad (18)$$

$A = 0$ and

$$\begin{aligned} C &= \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \overset{\diamond}{\text{curl}}_\eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & - \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \overset{\diamond}{\text{curl}}_\eta \\ \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \overset{\diamond}{\text{curl}}_\eta & 0 \end{pmatrix} \end{aligned}$$

continuous on $\overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)}$.

$$\begin{aligned} \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \overset{\diamond}{\text{curl}}_\eta &= \eta^{-1} \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \left(\eta \overset{\diamond}{\text{curl}}_\eta + 1 \right) - \eta^{-1} \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \\ &= \eta^{-1} - \eta^{-1} \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \end{aligned}$$

From the abstract theory we now find existence, uniqueness of a solution in the space $H_{\ell,0} \left(\mathbb{R}, \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}} \right)} \right)$ and continuous (causal) dependence on the right-hand side. Moreover,

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$$\pi_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \pi_\eta^* \begin{pmatrix} e \\ h \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)^{-1} W_0 \in H_{\varrho,1} \left(\mathbb{R}, \overline{R \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)} \right).$$

With

$$\begin{pmatrix} E \\ H \end{pmatrix} := \pi_\eta^* \begin{pmatrix} e \\ h \end{pmatrix} = \pi_\eta^* \pi_\eta \pi_\eta^* \begin{pmatrix} e \\ h \end{pmatrix}$$

and using that $P_\eta = \pi_\eta^* \pi_\eta$ commutes with $\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$ we get

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \pi_\eta^* \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)^{-1} W_0 \in H_{\varrho,1} \left(\mathbb{R}, L^2(\Omega) \right).$$

Applying $\left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)$ now yields

$$\begin{aligned} & \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \\ & - \chi_{\mathbb{R}_{>0}} \otimes \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \pi_\eta^* \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)^{-1} W_0 \in H_{\varrho,1} \left(\mathbb{R}, H_{-1} \left(\overset{\diamond}{\text{curl}} + \text{i} \right) \right) \end{aligned}$$

Clearly,

$$\begin{aligned} \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \pi_\eta^* \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)^{-1} W_0 &= \pi_\eta \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \pi_\eta^* \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)^{-1} W_0 \\ &= \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right)^{-1} W_0 \\ &= W_0 \end{aligned}$$

and so

$$\left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes W_0 \in H_{\varrho,1} \left(\mathbb{R}, H_{-1} \left(\overset{\diamond}{\text{curl}} + \text{i} \right) \right).$$

The latter shows by causality and a temporal Sobolev embedding argument that

$$t \mapsto \left(\left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes W_0 \right) (t)$$

is continuous on \mathbb{R} and in particular

$$\left(\left(1 + \eta \overset{\diamond}{\text{curl}}_\eta\right) \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes W_0 \right) (0) = 0.$$

The latter implies

$$\begin{pmatrix} D \\ B \end{pmatrix} (0+) = W_0.$$

Applying $\left(1 + \eta \overset{\diamond}{\text{curl}}\right) \pi_\eta^*$ to equation (17) yields similarly

$$\partial_0 \begin{pmatrix} D \\ B \end{pmatrix} + \overset{\diamond}{\text{curl}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathcal{J} + \delta \otimes W_0,$$

which is the equation of the Drude-Born-Fedorov model. Note that this equation now holds in $H_{\varrho,-1}(\mathbb{R}, H_{-1}(\overset{\diamond}{\text{curl}} + i))$. Causality and continuity estimates follow from the general theory applied to (17).

Conversely, if $\begin{pmatrix} E \\ H \end{pmatrix}$ solves (15) with (16) with $\mathcal{J} = 0$ and $W_0 = 0$, then

$$\left(\partial_0 \left(1 + \eta \overset{\diamond}{\text{curl}} \right) \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \overset{\diamond}{\text{curl}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = 0 \quad (19)$$

and so

$$\left(1 + \eta \overset{\diamond}{\text{curl}} \right) \left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \eta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = \eta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}. \quad (20)$$

Thus, we have that

$$\eta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \in R \left(1 + \eta \overset{\diamond}{\text{curl}} \right).$$

This observation implies

$$\left(\partial_0 P_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + P_\eta \eta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \eta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}$$

or

$$\left(\partial_0 P_\eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \overset{\diamond}{\text{curl}}_\eta \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = 0.$$

This implies

$$\begin{aligned} 0 &= \Re \left\langle P_\eta \begin{pmatrix} E \\ H \end{pmatrix} \middle| \left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \overset{\diamond}{\text{curl}}_\eta \left(1 + \eta \overset{\diamond}{\text{curl}}_\eta \right)^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) P_\eta \begin{pmatrix} E \\ H \end{pmatrix} \right\rangle_{\varrho,0,0} \\ &= \Re \left\langle P_\eta \begin{pmatrix} E \\ H \end{pmatrix} \middle| \partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} P_\eta \begin{pmatrix} E \\ H \end{pmatrix} \right\rangle_{\varrho,0,0} \\ &= \varrho \left\langle P_\eta \begin{pmatrix} E \\ H \end{pmatrix} \middle| \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} P_\eta \begin{pmatrix} E \\ H \end{pmatrix} \right\rangle_{\varrho,0,0} \end{aligned}$$

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and we see

$$P_\eta \begin{pmatrix} E \\ H \end{pmatrix} = 0.$$

On the other hand, we read off from (20) that

$$\begin{aligned} 0 &= (1 - P_\eta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1 - P_\eta) \begin{pmatrix} E \\ H \end{pmatrix} \end{aligned}$$

implying

$$(1 - P_\eta) \begin{pmatrix} E \\ H \end{pmatrix} = 0.$$

This shows $\begin{pmatrix} E \\ H \end{pmatrix} = 0$, i.e. uniqueness. □

Remark 2.2. Since the known results on the spectrum of $\overline{\text{curl}^\diamond}$ show compact resolvent in the ortho-complement of the null space of curl^\diamond , we also have $R\left(1 + \eta \overline{\text{curl}^\diamond}\right) = R\left(1 + \eta \text{curl}^\diamond\right)$ in these cases. If we have $-\frac{1}{\eta} \in \rho\left(\text{curl}^\diamond\right)$ the result becomes particularly simple. In this case we always have $\overline{R\left(1 + \eta \text{curl}^\diamond\right)} = R\left(1 + \eta \text{curl}^\diamond\right) = L^2(\Omega)$, which eliminates the possibly undesirable range conditions (13), (14).

Note that the assumption of Theorem 2.1 may never hold (making the claim of the theorem trivially correct), as e.g. in the exterior domain case. However, for example in the case of Ω having bounded measure, we have – according to [5] – pure point spectrum with no accumulation points and so the assumption of Theorem 2.1 is non-trivial and says simply $-\frac{1}{\eta} \in \mathbb{R}$ or

$$\eta \in \mathbb{R} \setminus \{0\}.$$

In the case $\eta = 0$, although not covered by the theorem, the solution theory is just the standard regular case.

3 A Note on Generalizations of the Drude-Born-Fedorov Model

By virtue of the power of the theoretical framework the above solution strategies can be conveniently generalized. We may straightforwardly generalize the Drude-Born-Fedorov

model (16) by imposing instead

$$\begin{pmatrix} D \\ B \end{pmatrix} = \left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right) M_* (\partial_0^{-1}) \begin{pmatrix} E \\ H \end{pmatrix} \quad (21)$$

where

$$\kappa (\partial_0^{-1}) = \kappa_0 + \partial_0^{-1} \kappa_1 (\partial_0^{-1})$$

and

$$M_* (\partial_0^{-1}) = M_{*,0} + \partial_0^{-1} M_{*,1} (\partial_0^{-1})$$

with $\kappa_0, M_{*,0}$ selfadjoint, continuous, strictly positive definite and κ, M_* bounded, analytic in $B_{\mathbb{C}}(r, r)$, for some $r \in \mathbb{R}_{>0}$. It should be noted that convolution integral type generalizations of the Drude-Born-Fedorov model have been previously suggested in [12], but discussed only for the time-harmonic case. Also, the particular model based on homogenization with $\kappa (\partial_0^{-1}) = \eta^{-1}$, $M_{*,1} = \eta \varepsilon k \times$, $M_{*,0} = \eta \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$, $k \in \mathbb{R}^3$, has been suggested in [21].

Assuming

$$-1 \in \rho \left(\kappa_0^{-1} \mathring{\text{curl}} \right)$$

as a replacement for assumption (12) we find that $\left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right)$ is boundedly invertible for all sufficiently large $\varrho \in \mathbb{R}_{>\frac{1}{2r}}$. Indeed, in this case

$$\begin{aligned} & \left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right)^{-1} = \\ & = \left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right)^{-1} \\ & = \left(\kappa_0 - \partial_0^{-1} \kappa_1 (\partial_0^{-1}) + \mathring{\text{curl}} \right)^{-1} \\ & = (1 - Q_0 (\partial_0^{-1}))^{-1} \left(\kappa_0 + \mathring{\text{curl}} \right)^{-1} \end{aligned}$$

with

$$Q_0 (\partial_0^{-1}) := \left(\kappa_0 + \mathring{\text{curl}} \right)^{-1} \partial_0^{-1} \kappa_1 (\partial_0^{-1})$$

as an ad-hoc abbreviation. Moreover,

$$\begin{aligned} & \left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right)^{-1} \mathring{\text{curl}} = \\ & = (1 - Q_0 (\partial_0^{-1}))^{-1} \left(\kappa_0 + \mathring{\text{curl}} \right)^{-1} \mathring{\text{curl}} \end{aligned}$$

3 A Note on Generalizations of the Drude-Born-Fedorov Model

shows that

$$C = \left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right)^{-1} \mathring{\text{curl}}$$

is a bounded operator. So, with our assumption $-1 \in \rho \left(\kappa_0 \mathring{\text{curl}} \right)$, as a by-product of the proof of Theorem 2.1, the well-posedness result extends also to this generalization. The modified material law for the analogue of (17) is

$$M (\partial_0^{-1}) = M_* (\partial_0^{-1}) + \partial_0^{-1} C.$$

Note that the initial condition now assumes the form

$$\left(\left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right) M_* (\partial_0^{-1}) \begin{pmatrix} E \\ H \end{pmatrix} \right) (0+) = \left(\kappa_0 + \mathring{\text{curl}} \right) M_{0,*} \begin{pmatrix} E \\ H \end{pmatrix} (0+) = W_0.$$

In summary, we obtain as a simple corollary to the proof of Theorem 2.1 (taking Remark 2.2 into account):

Corollary 3.1. *Let **Hypothesis Ω** hold and let*

$$\kappa (\partial_0^{-1}) = \kappa_0 + \partial_0^{-1} \kappa_1 (\partial_0^{-1})$$

and

$$M_* (\partial_0^{-1}) = M_{*,0} + \partial_0^{-1} M_{*,1} (\partial_0^{-1})$$

with $\kappa_0, M_{*,0}$ selfadjoint, continuous, strictly positive definite and κ, M_* bounded, analytic in $B_{\mathbb{C}}(r, r)$, for some $r \in \mathbb{R}_{>0}$. Then we have that for every

$$-\frac{1}{\eta} \in \rho \left(\mathring{\text{curl}} \right) \cap \mathbb{R}, \quad (22)$$

$$\mathcal{J} \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega)) \quad (23)$$

and

$$W_0 \in L^2(\Omega) \quad (24)$$

the Drude-Born-Fedorov model

$$\partial_0 \begin{pmatrix} D \\ B \end{pmatrix} + \begin{pmatrix} 0 & -\mathring{\text{curl}} \\ \mathring{\text{curl}} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathcal{J} + \delta \otimes W_0 \quad (25)$$

with

$$\begin{pmatrix} D \\ B \end{pmatrix} = \left(\kappa (\partial_0^{-1}) + \mathring{\text{curl}} \right) M_* (\partial_0^{-1}) \begin{pmatrix} E \\ H \end{pmatrix} \quad (26)$$

has a unique solution $\begin{pmatrix} E \\ H \end{pmatrix} \in H_{\rho,0}(\mathbb{R}, L^2(\Omega))$ satisfying

$$\begin{pmatrix} D \\ B \end{pmatrix}(0+) = W_0$$

in $H_{-1}\left(\overset{\circ}{\text{curl}} + i\right)$. Moreover, the solution depends continuously and causal on the right-hand side data.

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